

Phase space geometry for constrained lagrangian systems

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Abstract

We study geometry of the phase space for finite dimensional dynamical systems with degenerate Lagrangians. The Lagrangian and Hamiltonian constraint formalisms are treated as the different local-coordinate pictures of the same invariant procedure. The invariant description is given in terms of geometrical objects associated with the structure of foliation on the phase space.

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I. INTRODUCTION

Since the appearance of the fundamental work of Dirac [1], a certain progress has been made in understanding geometry which underlies constrained dynamics [3] – [13]. For a system with regular (nondegenerate) Lagrangian an appropriate phase space in the case of Lagrangian (Hamiltonian) formalism is the tangent bundle TQ (cotangent bundle T^*Q) of the configurational space Q . In consideration of degenerate Lagrangians it is traditionally assumed that the phase space of the system is a subspace of TQ (Lagrangian formalism) or T^*Q (Hamiltonian formalism). Tangent and cotangent bundles are treated separately within the framework of the constrained dynamics, being related by the standard Legendre transformation. In this paper another point of view is advocated.

Given a Lagrangian $l \in C^\infty(TQ)$, the action S can be written as a functional

$$S[s] = \int_X s^* \varphi \quad (1)$$

acting on the sections s of the bundle $TQ \otimes R \rightarrow R$, where $X \in R$ is a compact set, $\varphi = -Edt + \theta_l$, $\theta_l = \frac{\partial l}{\partial v} dq$ is Cartan - Liouville one-form and E is the energy of the system. The Legendre transformation $L: TQ \rightarrow T^*Q$ is a diffeomorphism for nondegenerate l , so the transition to the Hamiltonian formalism can be viewed as a change of variables in (1), s being the section of $T^*Q \otimes R \rightarrow R$, $\varphi = -hdt + \theta$, where $\theta = pdq$, $\theta_l = L^*\theta$ and h is the Hamiltonian, $E = L^*h$.

For degenerate Lagrangians the map L is in general neither surjection nor injection. In this case the pleasant symmetry between the Lagrangian and Hamiltonian formalisms is broken, since the image of TQ by L now is only a subspace of the cotangent bundle. If one wishes to proceed in the usual way, the machinery of Dirac - Bergmann algorithm [1, 2] is applicable either in local-coordinate approach or by using the invariant (coordinate-free) description in the spirit of Gotay - Nester's work [5, 6, 7].

The invariant description of the constrained Lagrangian dynamics begins with the basic

notion of foliation on the tangent bundle and should in principle end up with the well-defined procedure of the reduction of the system to the submanifold of the lower dimension (which itself must admit a tangent bundle structure) with no second class constraints left in the theory. The next step is to compare the result with that one of the reduction procedure in the Hamiltonian formalism and check that the reduced submanifolds are connected by the nondegenerate Legendre transformation. For the specific choice of the Lagrangian function the reduction procedure of that kind has been described in [21] by making use of the tangent bundle geometry formalism developed earlier in [20].

The Hamiltonian reduction is controlled by the standard symplectic structure on T^*Q given by the closed nondegenerate two-form $\omega = dp \wedge dq$, whereas the analog of ω in the Lagrangian formalism, $\omega_l = L^*\omega$, is degenerate for degenerate ℓ .

The usual treatment of Lagrangian dynamics starts from the invariant form of the Euler - Lagrange equations on TQ ,

$$\dot{f} = X_E f, \quad (2)$$

$f \in C^\infty(TQ)$, where for any $g \in C^\infty(TQ)$ X_g is defined by

$$i_{X_g} \omega_\ell = -dg. \quad (3)$$

One may note that the relation (3) also plays the key role in geometric quantization [18, 19].

In this paper we pay more attention to the action (1) itself. In the case of degenerate Lagrangians the bundle $TQ \otimes R \rightarrow R$ appears not to be an appropriate phase space and should be replaced by the bundle $W \otimes R \rightarrow R$, where $W = TQ \oplus T^*Q$ is a Whitney sum of TQ and T^*Q ¹. The equations which come from the variational principle can still be written in the form of (2) but the relation (3) must be deformed taking into account the structure of the foliation on the original phase space W . This modification allows us to include into the scheme those functions (such as the second class constraints) which are not constant along

¹After this work was completed, we learned that similar approach was previously considered (in local coordinates) by W. Kundt [25] and by R. Skinner and R. Rusk (using coordinate-free formulation)[26]

the leaves of foliation. Then the equivalence of the Lagrangian and Hamiltonian formalisms becomes apparent and the constraints in both pictures are in one-to-one correspondence with each other. For nondegenerate ℓ the original phase space W decouples trivially into TQ and T^*Q at the first step of the constraint algorithm procedure.

It follows that the most general situation one encounters in the invariant description of the classical dynamics is that of the presymplectic manifold rather than the symplectic one.

The organization of the paper is as follows. In Sec. II a brief review of the invariant formulation of the Lagrangian and Hamiltonian dynamics is given for the systems with nondegenerate Lagrangians. The degenerate case is treated in local coordinates in Sec.III and the invariant formulation is provided in Sec. IV. Several illustrative examples are given in Sec.V. The Appendix contains definitions of some geometric objects used.

II. NONDEGENERATE LAGRANGIANS

A. LAGRANGIAN FORMALISM

For a given Lagrangian $l(q, \dot{q})$ the action

$$S[q(t)] = \int \ell(q, \dot{q}) dt \quad (4)$$

with the appropriate boundary conditions can be deliberately changed to

$$S[q, v] = \int dt \left(\ell(q, v) - v^i \frac{\partial \ell}{\partial v^i} + \dot{q}^i \frac{\partial \ell}{\partial v^i} \right) dt \quad (5)$$

to be considered as a functional on the tangent bundle TQ of the configurational space (smooth manifold) Q . We denote the independent local coordinates on TQ as (q^i, v^i) , $i = 1, \dots, N$, $N = \dim Q$, $l(q, v) \in C^\infty(TQ)$.

The Lagrangian l is called nondegenerate (regular) if the matrix $\Gamma_{ij} = \partial^2 l / \partial v_i \partial v_j$ has maximal rank $R = N$. Otherwise the Lagrangian is called degenerate (singular). Note that R does not depend on the particular choice of the local coordinates.

The variational principle applied to (5) gives

$$\frac{\delta S}{\delta q^i} = \frac{\partial \ell}{\partial q^i} - \frac{\partial^2 \ell}{\partial v^k \partial q^i} v^k + \frac{\partial^2 \ell}{\partial v^k \partial q^i} \dot{q}^k - \frac{d}{dt} \left(\frac{\partial \ell}{\partial v^i} \right) = 0, \quad (6)$$

$$\frac{\delta S}{\delta v^i} = \Gamma_{ij}(\dot{q}^j - v^j) = 0. \quad (7)$$

For nondegenerate l the last equation implies $\dot{q}^i = v^i$. Then (7) gives the Euler - Lagrange equations. One can define a special vector field $X_E \in \mathcal{X}(TQ)$ corresponding to (6),

$$X_E = v^i \frac{\partial}{\partial q^i} + \Gamma^{ij} \alpha_j \frac{\partial}{\partial v^i}, \quad (8)$$

where

$$\alpha_j = \frac{\partial \ell}{\partial q^j} - \frac{\partial^2 \ell}{\partial v^k \partial q^j} v^k \quad (9)$$

and $\Gamma^{ik} \Gamma_{kj} = \delta_{ij}$ such that for any $f \in C^\infty(TQ)$ its time derivative is given by

$$\dot{f} = X_E f. \quad (10)$$

In particular, for energy $E = \frac{\partial \ell}{\partial v^i} v^i - \ell$ we have

$$\dot{E} = X_E E = 0. \quad (11)$$

The invariant description can be given as follows. Let Q be a finite-dimensional smooth manifold, (TQ, π, Q) – its tangent bundle, $\ell \in C^\infty(TQ)$. The Cartan - Liouville one-form $\theta_L \in \Lambda^1(TM)$ is defined by

$$\theta_L = d\ell \circ \tau, \quad (12)$$

where τ is a canonical type $(1, 1)$ tensor field on TQ [4, 23]. Then the closed nondegenerate (for nondegenerate l) two-form $\omega_L = d\theta_L$ provides TQ with the symplectic structure. In local coordinates

$$\theta_L = \frac{\partial \ell}{\partial v^i} dq^i, \quad (13)$$

$$\omega_L = \Gamma_{ij} dv^i \wedge dq^j + \frac{1}{2} M_{ij} dq^i \wedge dq^j, \quad (14)$$

where

$$M_{ij} = \frac{\partial^2 \ell}{\partial q^i \partial v^j} - \frac{\partial^2 \ell}{\partial v^i \partial q^j}.$$

For $f, g \in C^\infty(TQ)$ one can define the Lagrangian bracket by

$$\{f, g\}_L = \omega_L(X_f, X_g), \quad (15)$$

where the vector field $X_f \in \mathcal{X}(TM)$ is defined by

$$i_{X_f} \omega_L = -df. \quad (16)$$

Since ω_L is nondegenerate, the map $\Omega^1(TQ) \rightarrow \mathcal{X}(TQ)$ (16) is an isomorphism. In local coordinates we have

$$\{f, g\}_L = \Gamma^{ij} \left(\frac{\partial f}{\partial v^i} \frac{\partial g}{\partial q^j} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial v^j} \right) - M^{ij} \frac{\partial f}{\partial v^i} \frac{\partial g}{\partial v^j}, \quad (17)$$

where $\Gamma^{ik} \Gamma_{kj} = \delta_j^i$, $M^{ik} = \Gamma^{ik} M_{kl} \Gamma^{lj}$. This bracket is antisymmetric. It obeys Jacobi identity since $d\omega_L = 0$. The energy E can be introduced as a smooth function on TQ by

$$E = i_Z \theta_L - \ell, \quad (18)$$

where $Z = \{X \in \mathcal{X}(TQ) : \tau X = \Delta\}$, Δ is the Liouville vector field.

Putting all this together, we can write the action (5) in the invariant form (1). For the energy (18) one can define the corresponding vector field using (16). A simple local-coordinate calculation shows that this vector field is equivalent to the field X_E (8) which originates from the variational principle. Therefore for any $f \in C^\infty(TQ)$ we can write

$$\dot{f} = X_E f = df(X_E) = \omega_L(X_E, X_f) = \{E, f\}_L. \quad (19)$$

In particular,

$$\dot{E} = dE(X_E) = 0. \quad (20)$$

B. Hamiltonian formalism

The transition to the Hamiltonian form of dynamics is provided by the Legendre transformation $L : T_q Q \rightarrow T_q^* Q$

$$(q, v) \rightarrow (q, p), \quad p_i = \frac{\partial \ell}{\partial v^i}, \quad (21)$$

$(q, p) \in T^*Q$ which is a diffeomorphism for nondegenerate l . In the new variables the action (1) becomes

$$S[q, v] \rightarrow S[q, p],$$

$$S = \int_X s^* \varphi, \quad (22)$$

where $\varphi = -hdt + \theta$, $\varphi \in \Omega^1(T^*Q \otimes R)$, $X \subset R^1$, $h(q, p) = (L^*)^{-1}(E)$ is the Hamiltonian, $\theta = p_i dq^i$ is the canonical one-form on T^*Q , $\theta_L = L^*(\theta)$.

The variational principle applied to (22) gives the standard Hamilton's equations which can be associated with the vector field $X_h \in \mathcal{X}(T^*Q)$ (the counterpart of X_E (8)),

$$X_h = \frac{\partial h}{\partial p^i} \frac{\partial}{\partial q^i} - \frac{\partial h}{\partial q^i} \frac{\partial}{\partial p^i}. \quad (23)$$

The field X_h can be defined invariantly by

$$i_{X_h} \omega = -dh, \quad (24)$$

where $\omega = d\theta$ is the nondegenerate two-form which provides T^*Q with the standard symplectic structure. Then $\forall f \in C^\infty(T^*M)$ we have

$$\dot{f} = X_H f = df(X_H) = \omega(X_H, X_f) = \{H, f\}, \quad (25)$$

where $\{f, g\}$ is the Poisson bracket. In particular,

$$\dot{h} = dh(X_h) = 0. \quad (26)$$

III. DEGENERATE LAGRANGIANS

For degenerate Lagrangians the Legendre transformation (21) is not a bijection. This means in particular that $R = \text{rank } \Gamma_{ij} \leq N$ so the condition $\dot{x}^i = v^i$ is valid now only for a part of coordinates, i.e. for $\dot{x}^a = v^a$, $a=1, \dots, R$. The transition to the Hamiltonian formulation can no longer be considered as a change of variables in the action (1). To cure this situation one should extend the phase space by adding $N - R$ new independent variables p_μ ($\mu=1, \dots, N - R$) and write the action S as follows [15]

$$S[q, v, p_\mu] = \int dt \left(\ell(q, v) - v^a \frac{\partial \ell}{\partial v^a} + \dot{q}^a \frac{\partial \ell}{\partial v^a} + p_\mu (\dot{q}^\mu - v^\mu) \right) \quad (27)$$

which can also be presented in the form

$$S = \int \varphi_1, \quad (28)$$

with $\varphi_1 = -E_1 dt + \theta_1$, where

$$E_1 = v^a \frac{\partial \ell}{\partial v^a} + p_\mu v^\mu - \ell, \quad (29)$$

$$\theta_1 = \frac{\partial \ell}{\partial v^a} dq^a + p_\mu dq^\mu. \quad (30)$$

The variation of S gives the equations

$$\dot{v}^a = \Gamma^{ab} \frac{\partial \ell}{\partial q^b} - \Gamma^{ab} \frac{\partial^2 \ell}{\partial v^b \partial q^i} v^i - \Gamma^{ab} \Gamma_{b\mu} \dot{v}^\mu, \quad (31)$$

$$\dot{p}_\mu = \frac{\partial \ell}{\partial q^\mu}, \quad (32)$$

$$\dot{q}^i - v^i = 0, \quad (33)$$

$$p_\mu - \frac{\partial \ell}{\partial v^\mu} = 0. \quad (34)$$

The last equation is a constraint in the space of variables q^i, v^i, p_μ . The time derivative of a smooth function $f(q^i, v^i, p_\mu)$ can be written now as

$$\dot{f} = Y^{(1)} f + \dot{v}^\mu Y_\mu^{(1)} f, \quad (35)$$

$$\phi_\mu = p_\mu - \frac{\partial l}{\partial v^\mu} = 0, \quad (36)$$

where

$$Y^{(1)} = v^i \frac{\partial}{\partial q^i} + \Gamma^{ab} \alpha_b \frac{\partial}{\partial v^a} + \frac{\partial \ell}{\partial q^\mu} \frac{\partial}{\partial p^\mu}, \quad (37)$$

$$\alpha_b = \frac{\partial \ell}{\partial q^b} - \frac{\partial^2 \ell}{\partial v^b \partial q^i} v^i, \quad (38)$$

$$Y_\mu^{(1)} = b_\mu^i \frac{\partial}{\partial v^i}, \quad (39)$$

and b_μ^i are the null-vectors of Γ_{ij} .

The action (28), however, does not have an invariant form. For the sake of invariance one should write S as

$$S[q, v, p] = \int dt \left(l - p_i v^i + p_i \dot{q}^i \right). \quad (40)$$

Application of the variational principle gives dynamical equations

$$\dot{p}_i = \frac{\partial l}{\partial v^i}, \quad (41)$$

$$\dot{q}^i = v^i \quad (42)$$

as well as the *primary* constraints

$$\phi_i = p_i - \frac{\partial l}{\partial v^i} = 0. \quad (43)$$

The action (40) can be considered as the basic one for both the Lagrangian and the Hamiltonian formulations for either degenerate or nondegenerate Lagrangians.

Equations (41)-(42) imply that the evolution of any smooth function $f(q, v, p)$ is given by

$$\dot{f} = Y f + v^i Y_i f, \quad (44)$$

$$\phi_i = p_i - \frac{\partial l}{\partial v^i} = 0, \quad (45)$$

where

$$Y = v^i \frac{\partial}{\partial q^i} + \frac{\partial \ell}{\partial q^i} \frac{\partial}{\partial p^i}, \quad (46)$$

$$Y_i = \frac{\partial}{\partial v^i}. \quad (47)$$

Since $Y_i \phi_j = -\Gamma_{ij}$, the Dirac's compatibility condition $\dot{\phi}_i = 0$ leads to determination of R accelerations

$$\dot{v}^a = (Y \phi_b) \Gamma^{ba} - \dot{v}^\mu \Gamma_{\mu b} \Gamma^{ba} \quad (48)$$

as well as to the new constraints

$$\phi_\mu^{(1)} = b_\mu^i (Y \phi_i) = 0. \quad (49)$$

At the same time a part of constraints in (45), namely those for which

$$\det(Y_a \phi_b) \neq 0, \quad (50)$$

$Y_a = \frac{\partial}{\partial v^a}$, can be resolved. Because of the condition (50), we are free to choose either v^a or p_a as the local coordinates on the resulting surface $M_1 = \{q^i, p_i, v^i : \phi_a = 0, a = 1, \dots, R\}$. As a result of this choice one gets either Lagrangian or Hamiltonian formulation, correspondingly.

A. LAGRANGIAN FORMALISM

In the Lagrangian scheme the system (44)-(45) on M_1 becomes

$$\dot{f} = Y^{(1)} f + \dot{v}^\mu Y_\mu^{(1)} f, \quad (51)$$

$$\phi_\mu |_{M_1} = 0, \quad (52)$$

$$\phi_\mu^{(1)} |_{M_1} = 0, \quad (53)$$

where $Y^{(1)}$ and $Y_\mu^{(1)}$ have been defined in (37)-(39). The action S (40) confined to M_1 is given by (27). We will call the primary constraints for which (50) (and consequently (48)) holds *the primary constraints of the second class*. The rest of them are called the primary constraints of the first class.

Next the condition $\dot{\phi}_\mu^{(1)} = 0$ is to be verified. This gives R_1 determined accelerations,

$$\dot{v}^{a_1} = - \left(Y^{(1)} \phi_{b_1}^{(1)} \right) \gamma^{(1)b_1 a_1} - \dot{v}^{\mu_1} \left(Y_{\mu_1}^{(1)} \phi_{b_1}^{(1)} \right) \gamma^{(1)b_1 a_1}, \quad (54)$$

and a new set of constraints,

$$\phi_{\mu_1}^{(2)} = b_{\mu_1}^{(2)\mu} \left(Y^{(1)} \phi_{\mu}^{(1)} \right) = 0. \quad (55)$$

Here $\gamma_{\mu\nu}^{(1)} = Y_{\mu}^{(1)} \phi_{\nu}^{(1)}$, $R_1 = \text{rank } \gamma_{\mu\nu}^{(1)}$, $a_1 = 1, \dots, R_1$, $\mu_1 = 1, \dots, N - R - R_1$ and $b_{\mu_1}^{(2)\mu} = \delta_{\mu_1}^{\mu} - \delta_{a_1}^{\mu} \gamma^{a_1 b_1} \gamma_{b_1 \mu_1}$ are the null vectors of $\gamma_{\mu\nu}^{(1)}$. Again, among the secondary constraints (53) there are R_1 constraints of the second class ($\det \gamma_{ab} \neq 0$) as well as $N - R - R_1$ those of the first class. The system therefore can be reduced to the surface $M_2 \subset M_1$ of the second class secondary constraints $\phi_{a_1}^{(1)} = 0$ (the corresponding number of the primary constraints (R_1) must also be resolved, since $\dot{\phi}_{a_1} \sim \phi_{a_1}^{(1)}$). The evolution then is given by

$$\dot{f} = Y^{(2)} f + \dot{v}^{\mu_1} Y_{\mu_1}^{(2)} f, \quad (56)$$

$$\phi_{\mu_1} \big|_{M_2} = 0, \quad (57)$$

$$\phi_{\mu_1}^{(1)} \big|_{M_2} = 0, \quad (58)$$

$$\phi_{\mu_1}^{(2)} \big|_{M_2} = 0, \quad (59)$$

where

$$Y^{(2)} = \left(Y^{(1)} - \left(Y^{(1)} \phi_{b_1}^{(1)} \right) \gamma^{(1) b_1 a_1} Y_{a_1}^{(1)} \right) \big|_{M_2}, \quad (60)$$

$$Y_{\mu_1}^{(2)} = b_{\mu_1}^{(2)\mu} Y_{\mu}^{(1)} \big|_{M_2}. \quad (61)$$

The following iterations are straightforward. After the k -th step the system becomes

$$\dot{f} = Y^{(k)} f + \dot{v}^{\mu_{k-1}} Y_{\mu_{k-1}}^{(k)} f, \quad (62)$$

plus the set of constraints

$$\phi_{\mu_{k-1}}^{\alpha} \big|_{M_k} = 0, \quad (63)$$

where $\alpha = 0, 1, \dots, k$, $\phi_{\mu_{k-1}}^{(0)} = \phi_{\mu_{k-1}}$,

$$Y^{(k)} = \left(Y^{(k-1)} - \left(Y^{(k-1)} \phi_{b_{k-1}}^{(k-1)} \right) \gamma^{(k-1) b_{k-1} a_{k-1}} Y_{a_{k-1}}^{(k-1)} \right) \big|_{M_k}, \quad (64)$$

$$M_k = \left\{ x \in M_{k-1} : \phi_{a_{k-1}}^{(\alpha)}(x) = 0, \alpha = 0, \dots, k \right\}, M_k \subset M_{k-1} \subset M_{k-2} \subset \dots \subset W.$$

The iteration process terminates, if:

1) for certain k we have

$$R_k = N - R_1 - \dots - R_{k-1},$$

where $R_k = \text{rank } \gamma_{b_{k-1}a_{k-1}}^{(k)}$. In this case all the accelerations are determined and the dynamics confined to the final constraint surface M_k is totally fixed.

2) all the constraints of generation $k+1$ are reducible to those of the previous generations or are identically zero. In this case $N - R_1 - \dots - R_k$ accelerations remain undetermined and the system possesses certain “gauge freedom”.

B. HAMILTONIAN FORMALISM

Let us rewrite the system (44) - (45) in a slightly different form

$$\dot{f} = \{E, f\} + \dot{v}^i Y_i f, \quad (65)$$

$$\phi_i = 0, \quad (66)$$

where $E(p, q, v) = p_i v^i - l$ and $\{f, g\}$ is the usual Poisson bracket generated by the two-form $dp \wedge dq$ (which is of course degenerate in the space of variables p, q, v). As it was in (54) - (55), we determine R accelerations,

$$\dot{v}^a = \{E, \phi_b\} \Gamma^{ba} - \dot{v}^\mu \Gamma_{\mu b} \Gamma^{ba}, \quad (67)$$

and a set of new constraints,

$$\phi_\mu^{(1)} = b_\mu^i \{E, \phi_i\} = 0. \quad (68)$$

The condition (50) allows us to resolve a certain part of constraints in (66). To produce the Hamiltonian scheme, one should choose p_a rather than v^a as a set of local coordinates on the surface M_1 of the resolved constraints. Due to the identity $\Gamma_{\mu\nu} = \Gamma_{\mu a} \Gamma^{ab} \Gamma_{b\nu}$ the Routh's function

$$R(q^i, p_a, v^\mu) = (l - p_a v^a) |_{M_1} \quad (69)$$

is linear in v^μ :

$$R = -h(q^i, p_a) + v^\mu \psi_\mu(q^i, p_a) \quad (70)$$

(here $\psi_\mu(q^i, p_a) = \partial l / \partial v^\mu \mid_{M_1}$), so the action (40) being confined to M_1 in coordinates p_a, q^i, v^μ reads

$$S[q^i, p_a, v^\mu] = \int dt \left(-h_T + p_i \dot{q}^i \right), \quad (71)$$

where

$$h_T = h(q^i, p_a) + v^\mu \phi_\mu, \quad (72)$$

$$\phi_\mu = p_\mu - \psi_\mu(q^i, p_a).$$

Correspondingly, the system (65) - (66) becomes

$$\dot{f} = \{h, f\} + v^\mu \{\phi_\mu, f\} + \dot{v}^\mu Y_\mu f, \quad (73)$$

$$\phi_\mu \mid_{M_1} = 0, \quad (74)$$

$$\phi_\mu^{(1)} \mid_{M_1} = 0, \quad (75)$$

Note that (28) and (71) are equivalent. The transition between the Lagrangian and the Hamiltonian formulations is a smooth change of variables in the action, as it was for non-degenerate Lagrangians. There is no need to add voluntarily the primary constraints ϕ_μ to the function $h(q^i, p_a)$, since the old Dirac's construction h_T naturally appears in (71) and in (73).

The condition $\dot{\phi}_\mu^{(1)}=0$ leads to the determination of some accelerations,

$$\dot{v}^{a_1} = - \left\{ h_T, \phi_{b_1}^{(1)} \right\} \gamma^{b_1 a_1} - \dot{v}^{\mu_1} \gamma_{\mu_1 b_1} \gamma^{b_1 a_1}, \quad (76)$$

and to the new constraints,

$$\phi_{\mu_1}^{(2)} = b_{\mu_1}^{(2)\mu} \left\{ h_T, \phi_{\mu_1}^{(1)} \right\} = 0. \quad (77)$$

Here $\gamma_{\mu\nu} = \partial(\phi_\nu^{(1)} \mid_{M_1}) / \partial v^\mu$. Now we can reduce the system to the surface $M_2 \subset M_1$ of the second class secondary constraints $\phi_{a_1}^{(1)} = 0$. In the Hamiltonian scheme we can do this

explicitly by expressing v^{a_1} :

$$v^{a_1} = -\{h, \phi_{b_1}\} \gamma^{b_1 a_1} - v^{\mu_1} \{\phi_{\mu_1}, \phi_{b_1}\} \gamma^{b_1 a_1}. \quad (78)$$

Then the system (73)-(75) on M_2 can be written as

$$\dot{f} = \{h, f\}_1^* + v^\mu \{\phi_{\mu_1}, f\}_1^* + \dot{v}^{\mu_1} Y_{\mu_1} f, \quad (79)$$

$$\phi_{\mu_1} |_{M_2} = 0, \quad (80)$$

$$\phi_{\mu_1}^{(1)} |_{M_2} = 0, \quad (81)$$

$$\phi_{\mu_1}^{(2)} |_{M_2} = 0 \quad (82)$$

where

$$\{f, g\}_1^* = \{f, g\} - \{f, \phi_{b_1}\} \gamma^{b_1 a_1} \{\phi_{a_1}, g\} \quad (83)$$

is the Dirac bracket in respect to the second class primary constraints. Next iteration gives us another set of the determined accelerations,

$$\dot{v}^{a_2} = -\{h, \phi_{b_2}^{(2)}\}_1^* \gamma^{(2) b_2 a_2} - v^{\nu_1} \{\phi_{\nu_1}, \phi_{b_2}^{(2)}\}_1^* \gamma^{(2) b_2 a_2} - \dot{v}^{\mu_2} \gamma_{\mu_2 b_2}^{(2)} \gamma^{b_2 a_2}, \quad (84)$$

velocities,

$$v^{a_2} = -\{h, \phi_{b_2}^{(1)}\}_1^* \gamma^{(2) b_2 a_2} - v^{\mu_2} \{\phi_{\mu_2}, \phi_{b_2}^{(1)}\}_1^* \gamma^{(2) b_2 a_2}, \quad (85)$$

where

$$\gamma_{\mu_1 \nu_1}^{(2)} = \partial \phi_{\nu_1}^{(2)} / \partial v^{\mu_1} = \{\phi_{\mu_1}, \phi_{\nu_1}^{(1)}\}_1^*, \quad (86)$$

and constraints

$$\phi_{\mu_2}^{(3)} = b_{\mu_2}^{(3) \mu_1} \left(\{h, \phi_{\mu_1}^{(2)}\}_1^* + v^{\nu_1} \{\phi_{\nu_1}, \phi_{\mu_1}^{(2)}\}_1^* \right) = 0. \quad (87)$$

The system (79)-(82) then becomes

$$\dot{f} = \{h, f\}_2^* + v^{\mu_2} \{\phi_{\mu_2}, f\}_2^* + \dot{v}^{\mu_2} Y_{\mu_2} f, \quad (88)$$

$$\phi_{\mu_2} |_{M_3} = 0, \quad (89)$$

$$\phi_{\mu_2}^\alpha|_{M_3} = 0, \alpha = 1, 2, 3, \quad (90)$$

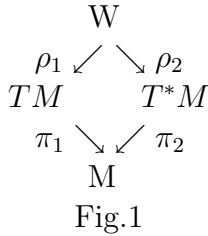
where

$$\{f, g\}_2^* = \{f, g\}_1^* - \left\{f, \phi_{b_2}^{(1)}\right\}_1^* \gamma^{(2)b_2 a_2} \{\phi_{a_2}, g\}_1^* + \left\{g, \phi_{b_2}^{(1)}\right\}_1^* \gamma^{(2)b_2 a_2} \{\phi_{a_2}, f\}_1^*. \quad (91)$$

The iterative procedure continues until either all acceleration (and velocities) are determined (and the final phase space is a submanifold of T^*Q) or the constraints of the k-th generation are reducible to the previous ones (or are identically zero).

IV. INVARIANT FORMULATION

Let M be a real smooth N -dimensional manifold, (TM, π_1, M) its tangent bundle, (T^*M, π_2, M) its cotangent bundle. Let $W = TM \oplus T^*M$ denotes a Whitney sum of these two bundles (Fig.1).



Consider $\rho_2^* \theta \in \Omega^1(W)$, where θ is the canonical one-form on T^*M . To simplify notations we denote $\rho_2^* \theta$ by θ again. The two-form $\omega = d\theta$ is closed and degenerate on W , $\text{Ker } \omega = \mathcal{X}^V(TM) \subset \mathcal{X}(W)$. The elements of $\text{Ker } \omega$ form an integrable distribution \mathcal{D} on W thus defining a foliation F of W , the leaves of F being the maximal integral manifolds of \mathcal{D} . The foliation F has codimension $q = 3N - \text{rank } \mathcal{D} = \dim \text{Coker } \omega = 2N$. We assume that the space of leaves \mathcal{L} has a structure of manifold, so F is a regular foliation [11] of codimension q . Now consider the de Rham complex of F . We have $\Omega^1 W / \mathcal{L} = T^*W / \mathcal{L}$, $\Omega^i W / \mathcal{L} = \Lambda^i \Omega^1 W / \mathcal{L}$, $\Omega W / \mathcal{L} = \bigoplus_{i \geq 0} \Omega^i W / \mathcal{L}$. The first order differential operator $d_q: \Omega W / \mathcal{L} \rightarrow \Omega W / \mathcal{L}$ has the following properties [22]:

- i) $d_q: C^\infty(W) \rightarrow \Omega^1 W / \mathcal{L}$ is a composition $C^\infty(W) \xrightarrow{d} \Omega^1 W \rightarrow T^*W / \mathcal{L}$,

$$\text{ii) } d_q(\omega^p \wedge \omega^s) = d_q \omega^p \wedge \omega^s + (-1)^p \omega^p \wedge d_q \omega^s, \omega^p \in \Omega^p W / \mathcal{L},$$

$$\text{iii) } d_q^2 = 0.$$

Note also that $d_0 = d : \Omega W \rightarrow \Omega W$. In any Frobenius neighborhood $U \subset W$ one can choose the local coordinate system $(x^1, \dots, x^d, y^{d+1}, \dots, y^{d+q})$ such that \mathcal{D}_U is generated by the set $\{\partial/\partial x^\mu\}$, $\mu=1, \dots, d=\text{rank } \mathcal{D}$. Then the action of d_q on any $\alpha \in \Omega^p W / \mathcal{L}$,

$$\alpha = a_{\mu_1 \dots \mu_p}(x, y) dx^1 \wedge \dots \wedge dx^{\mu_p},$$

can be described as follows

$$d_q \alpha = \frac{\partial a_{\mu_1 \dots \mu_p}(x, y)}{\partial x^\nu} dx^\nu \wedge dx^1 \wedge \dots \wedge dx^{\mu_p},$$

$\mu, \nu=1, \dots, d$. In other words, provided $\{K_i\}_{i=1}^d$ is a basis in \mathcal{D} and $\{\theta^i\}_{i=1}^d$ is a dual basis in \mathcal{D}^* , the action of d_q is given by

$$d_q \alpha = (K_i a_{i_1 \dots i_p}) \theta^i \wedge \theta^1 \wedge \dots \wedge \theta^{i_p}.$$

Let us denote $\omega_f^q = d_q f$, $f \in C^\infty(W)$. Now we are ready to proceed with the invariant formulation of dynamics.

Let $Z = \{X \in \mathcal{X}(W) : \tau(\rho_{1*} X) = \Delta\}$, where τ is the vertical endomorphism, Δ is the Liouville field. Then for any Lagrangian $l \in C^\infty(TM)$ the energy $E \in C^\infty(W)$ is defined as follows

$$E = i_Z \theta - \rho_1^* l. \quad (92)$$

The action is a functional on the smooth sections of $W \otimes R \rightarrow R$,

$$S[s] = \int_X s^* \varphi, \quad (93)$$

where $X \in R$ is a compact set, $\varphi = -Edt + \theta$, $\varphi \in \Omega^1(W \otimes R)$. Since W is only a presymplectic manifold, the relation (16) between one-forms and vector fields on W now makes sense only for functions constant along the leaves of foliation F . In particular, (16) cannot be used for the second class constraints of any generation. However, in the local

coordinate approach to the constrained dynamics we do not encounter such restriction. To cure this, we deform the correspondence (16) and put it to be

$$i_{X_f}\omega = -df + \omega_f^q, \quad (94)$$

where q is the codimension of the foliation generated by $\text{Ker } \omega$. Then the system of equations (44) - (45) which follows from the variation of S can be written in the form

$$\dot{f} = df(X_E), \quad (95)$$

$$\omega_E^q = 0, \quad (96)$$

where X_E is defined by (94). Using (94) we can rewrite (95) as

$$\dot{f} = \omega(X_E, X_f) + \omega_f^q(X_E). \quad (97)$$

Note that (94) - (95) give $\dot{E} = 0$. One can decompose X_E as

$$X_E = Y + K, \quad (98)$$

where $K \in \mathcal{D}$ is an arbitrary vector. Locally, $K = \alpha^i K_i$, where $\{K_i\}_{i=1}^d$ is a basis in \mathcal{D} . Since $\mathcal{D} \in \mathcal{X}^V(TW)$, the yet undetermined multipliers α^i are interpreted as accelerations \dot{v}^i in any given chart. The equation (96) then reads

$$\omega_E^q = (K_i E) \theta^i = 0, \quad (99)$$

with $\theta^i(K_i) = \delta_j^i$. This defines a subspace $S \subset W$ by

$$\phi_i = dE(K_i) = 0. \quad (100)$$

For compatibility of (95) - (96) one must have therefore $X_E \in TS$, i.e.

$$d\phi_i(X_E) = 0. \quad (101)$$

If S is transversal to the leaves of F (in this case $\mathcal{D} \cap TS = \emptyset$ and $\text{rank } |d\phi_i(K_j)| = N$), the vector field X_E is completely fixed by the condition (101) in the sense that all coefficients

α^i are determined. Then we can confine the system to S and choose either p's or v's as the set of local coordinates on S thus generating either Hamiltonian ($S = T^*Q$) or Lagrangian ($S = TQ$) form of dynamics.

In the generic case, however, $\mathcal{D} \cap TS = \mathcal{D}_1 \neq \emptyset$, $\text{rank } |d\phi_i(K_j)| = R_1 \leq N$, so only R_1 out of N accelerations are determined by (101). Instead of S we can choose now an intermediate transverse subspace M_1 ($S \subset M_1 \subset W$) such that $d_1 = \dim(\mathcal{D} \cup TM_1) = N - R_1$. The two-form $\omega_1 = \omega|_{M_1}$ is degenerate, $\dim \text{Ker } \omega_1 = d_1$. Now the scheme repeats itself. The system (95) – (96) becomes

$$\dot{f} = df(X_{E_1}), \quad (102)$$

$$\omega_E^q|_{M_1} = 0, \quad (103)$$

$$\omega_{E_1}^{q_1}|_{M_1} = 0, \quad (104)$$

where $E_1 = E|_{M_1}$, $q_1 = 2N$ is a codimension of the foliation of M_1 generated by ω_1 . This system can be expressed equivalently in terms of either Lagrangian ((51) – (53)) or Hamiltonian ((73) – (75)) local coordinates. A new subspace $S_1 \subset M_1$ defined by (103) arises here. The requirement $X_{E_1} \subset TS_1$ gives $R_2 \leq N - R_1$ determined accelerations. We choose M_2 such that $d_2 = \dim(\mathcal{D} \cap TM_2) = N - R_1 - R_2$. Now $\omega_2 = \omega_1|_{M_2}$ is degenerate and $\dim \text{Ker } \omega_2 = d_2$. The dynamics is governed by the system

$$\dot{f} = df(X_{E_2}) = \omega_2(X_{E_2}, X_f) + \omega_f^{q_2}(X_{E_2}), \quad (105)$$

$$\omega_E^q|_{M_2} = 0, \quad (106)$$

$$\omega_{E_1}^{q_1}|_{M_2} = 0, \quad (107)$$

$$\omega_{E_2}^{q_2}|_{M_2} = 0, \quad (108)$$

The sequence stops when for certain n S_n is not foliated, i.e. when $\mathcal{D}_{n-1} \cap TM_{n-1} = \emptyset$ and ω_n is nondegenerate. In this case the dynamics on $S_n = M_n \subset W$ is fixed and one has

$$\dot{f} = \omega_n(X_{E_n}, X_f) \quad (109)$$

for any $f \in C^\infty(S_n)$. As an alternative, one may discover, however, that for some n the condition $X_{E_{n-1}} \in TS_{n-1}$ is automatically satisfied while S_{n-1} is still foliated. This means that $N - R_1 - \dots - R_{n-1}$ accelerations remain undetermined and the evolution is described by the equation

$$\dot{f} = \omega_n(X_{E_n}, X_f) + \omega_f^{q_n}(X_{E_n}), \quad (110)$$

together with the final set of the first-class constraints

$$\omega_E^q|_{M_n} = 0, \dots, \omega_{E_n}^{q_n}|_{M_n} = 0. \quad (111)$$

The invariant geometrical treatment of the first-class constraints case including gauge-fixing procedure was given by L.D. Faddeev [3].

V. EXAMPLES

Example 1: Let us consider the simplest nondegenerate Lagrangian,

$$l = \frac{v^2}{2} - U(q). \quad (112)$$

We start with the phase space $W = TR^1 \oplus T^*R^1$. The canonical two-form $\omega = dp \wedge dq$ is degenerate on W , $\text{Ker } \omega = \text{span}\{\partial/\partial v\}$. The energy $E \in C^\infty(W)$ (92) is given by

$$E = pv - \frac{v^2}{2} + U(q). \quad (113)$$

The primary constraint one-form ω_E can be written as

$$\omega_E = d_q E = \frac{\partial E}{\partial v} dv = \phi dv = 0, \quad (114)$$

where $\phi = p - v$. The relation (94) gives the evolution vector field X_E on W ,

$$X_E = v \frac{\partial}{\partial q} - \frac{\partial U}{\partial q} \frac{\partial}{\partial p} + \dot{v} \frac{\partial}{\partial v}. \quad (115)$$

Thus the dynamics on W is given by $\{\dot{f} = X_E f, \phi = 0\}$. The compatibility condition (101) allows to determine the acceleration $\dot{v} = \partial U / \partial q$. This means that ϕ is the primary constraint

of the second class. Therefore, we reduce the system to $S_1 = \{x \in W : \phi = 0\}$. One can choose either (q, v) or (q, p) as the local coordinates on S_1 . In the first case one gets the Lagrangian scheme,

$$E_1 = E|_{S_1} = \frac{v^2}{2} + U(q), \quad (116)$$

$$X_{E_1} = v \frac{\partial}{\partial q} - \frac{\partial U}{\partial q} \frac{\partial}{\partial v}, \quad (117)$$

in the second case - the Hamiltonian description,

$$h = E|_{S_1} = \frac{p^2}{2} + U(q), \quad (118)$$

$$X_h = p \frac{\partial}{\partial q} - \frac{\partial U}{\partial q} \frac{\partial}{\partial p}. \quad (119)$$

In this case W trivially decouples into TQ and T^*Q .

Example 2: Consider the Lagrangian (Ref. [14])

$$l = \frac{v_1^2}{2} - v_2 q_3. \quad (120)$$

The initial phase space is $W = TR^3 \oplus T^*R^3$ with local coordinates $q_i, p_i, v_i, i = 1, 2, 3$. The fundamental two-form $\omega = dp_i \wedge dq_i$ is degenerate on W , $\text{Ker } \omega = \text{span} \{\partial/\partial v_1, \partial/\partial v_2, \partial/\partial v_3\}$.

The energy $E \in C^\infty(W)$,

$$E = p_1 v_1 + p_2 v_2 + p_3 v_3 - \frac{1}{2} v_1^2 + v_2 q_3, \quad (121)$$

defines (by (94)) the evolution vector field $X_E = Y + K$, where

$$Y = v_i \frac{\partial}{\partial q_i} - v_2 \frac{\partial}{\partial p_3}, \quad (122)$$

$$K = \dot{v}_i \frac{\partial}{\partial v_i} \quad (123)$$

with yet undetermined multipliers \dot{v}_i . The one-form (96) of the primary constraints is given by

$$\omega_E = d_q E = \frac{\partial E}{\partial v_i} dv_i = \phi_i dv_i = 0, \quad (124)$$

where $\phi_1 = p_1 - v_1$, $\phi_2 = p_2 + q_3$, $\phi_3 = p_3$. The compatibility condition (101) determines one of the accelerations, $\dot{v}_1 = 0$, and produces two new constraints,

$$\phi_2^{(1)} = v_3 = 0, \quad \phi_3^{(1)} = v_2 = 0. \quad (125)$$

Thus the primary constraint ϕ_1 is of the second class and can be resolved. On the intermediate transverse subspace $M_1 = \{x \in W : \phi_1 = 0\}$ the evolution is given by

$$\dot{f} = v_i \frac{\partial f}{\partial q_i} - v_2 \frac{\partial f}{\partial p_3} + \dot{v}_2 \frac{\partial f}{\partial v_2} + \dot{v}_3 \frac{\partial f}{\partial v_3}, \quad (126)$$

together with the set of constraints

$$M_2 = \{\phi_2 = 0, \phi_3 = 0, \phi_2^{(1)} = 0, \phi_3^{(1)} = 0\}. \quad (127)$$

Since $\dot{\phi}_2^{(1)} = \dot{v}_3 = 0$, $\dot{\phi}_3^{(1)} = \dot{v}_2 = 0$, all the secondary constraints are of the second class and the final constrained submanifold $S_2 = M_2$. For any $f \in C^\infty(S_2)$ one has then $\dot{f} = v_1 \frac{\partial f}{\partial q_1}$. The dynamics is totally fixed.

To consider the Hamiltonian form of dynamics, one should choose p_1 instead of v_1 as the local coordinate on M_1 . Note that

$$E|_{M_1} = \frac{1}{2}p_1^2 + v_2 q_3 + p_2 v_2 + p_3 v_3 = h + v_2 \phi_2 + v_3 \phi_3. \quad (128)$$

The equation (126) can be written in the form

$$\dot{f} = \{h, f\} + v_2 \{\phi_2, f\} + v_3 \{\phi_3, f\} + \dot{v}_2 \frac{\partial f}{\partial v_2} + \dot{v}_3 \frac{\partial f}{\partial v_3}, \quad (129)$$

together with the set M_2 (127). The compatibility condition again gives $\dot{v}_2 = 0$, $\dot{v}_3 = 0$.

One can make the canonocal transformation, $P_1 = p_1$, $Q_1 = q_1$, $P_2 = p_2$, $Q_2 = q_2 + p_3$, $P_3 = \phi_3 = p_3$, $Q_3 = q_3 + p_2 = \phi_2$. In terms of these new variables $S_2 = \{P_2 = 0, Q_2 = 0, P_3 = 0, Q_3 = 0\}$. One has $h|_{S_2} = P_1^2$ and $\dot{f} = \{h, f\}$ for any $f \in C^\infty(S_2)$.

Example 3: Let $l = \frac{1}{2}(q_1 v_2 - q_2 v_1 - q_1^2 - q_2^2)$ (Ref. [24]). The two-form $\omega = dp_i \wedge dq_i$, $i = 1, 2$, is degenerate on $W = TR^2 \oplus T^*R^2$, $\text{Ker } \omega = \text{span } \{\partial/\partial v_1, \partial/\partial v_2\}$. The energy is

$$E = p_1 v_1 + p_2 v_2 - l \quad (130)$$

and the evolution vector field X_E is given by

$$X_E = v_1 \frac{\partial}{\partial q_1} + v_2 \frac{\partial}{\partial q_2} + \left(\frac{v_2}{2} - q_1\right) \frac{\partial}{\partial p_1} - \left(\frac{v_1}{2} - q_2\right) \frac{\partial}{\partial p_2} + \dot{v}_1 \frac{\partial}{\partial v_1} + \dot{v}_2 \frac{\partial}{\partial v_2}, \quad (131)$$

The set of primary constraints is defined by

$$\omega_E = d_q E = \phi_i dv_i = 0, \quad (132)$$

where $\phi_1 = p_1 + q_2/2$, $\phi_2 = p_2 - q_1/2$. The condition (101) gives only the new constraints, $\phi_1^{(1)} = v_2 - q_1 = 0$, $\phi_2^{(1)} = v_1 + q_2 = 0$. The next iteration ($\dot{\phi}_1^{(1)} = 0$, $\dot{\phi}_2^{(1)} = 0$), however, allows to determine $\dot{v}_1 = -v_2$ and $\dot{v}_2 = v_1$. Therefore, the final constrained submanifold is

$$S = \{\phi_1 = 0, \phi_2 = 0, \phi_1^{(1)} = 0, \phi_2^{(1)} = 0\}.$$

The dynamics on S is completely fixed and $\forall f \in C^\infty(S)$ we have $\dot{f} = v_1 \frac{\partial f}{\partial q_1} - q_1 \frac{\partial f}{\partial v_1}$.

The energy (130) has the form

$$E = \frac{1}{2} (q_1^2 + q_2^2) + v_1 \phi_1 + v_2 \phi_2. \quad (133)$$

Instead of (131), (132) we can write

$$\dot{f} = \{E, f\} + \dot{v}_1 \frac{\partial f}{\partial v_1} + \dot{v}_2 \frac{\partial f}{\partial v_2}, \quad (134)$$

$$\phi_1 = 0, \quad (135)$$

$$\phi_2 = 0, \quad (136)$$

$\forall f \in C^\infty(W)$. The reduction to S must be “canonical” in Hamiltonian formalism. The canonical transformation $P_1 = p_1 - q_2/2$, $Q_1 = q_1/2 + p_2$, $P_2 = p_2 - q_1/2 = \phi_2$, $Q_2 = q_2/2 + p_1 = \phi_1$ gives $S = \{P_2 = 0, Q_2 = 0\}$ and $h = E|_S = \frac{1}{2} (P_1^2 + Q_1^2)$. The evolution equation (134) on S becomes $\dot{f} = \{h, f\}$, $f \in C^\infty(S)$.

Example 4: Now consider the Lagrangian (Ref.[16])

$$l = \frac{1}{2} e^{q_2} v_1^2.$$

As it was in the previous example, $W = TR^2 \oplus T^*R^2$, $\text{Ker } \omega = \text{span } \{\partial/\partial v_1, \partial/\partial v_2\}$. The energy

$$E = p_1 v_1 + p_2 v_2 - \frac{1}{2} e^{q_2} v_1^2 \quad (137)$$

produces (by (94)) the evolution vector field

$$X_E = v_1 \frac{\partial}{\partial q_1} + v_2 \frac{\partial}{\partial q_2} + \frac{1}{2} e^{q_2} v_1^2 \frac{\partial}{\partial p_2} + \dot{v}_1 \frac{\partial}{\partial v_1} + \dot{v}_2 \frac{\partial}{\partial v_2} \quad (138)$$

and the primary constrained form

$$\omega_E = \phi_1 dv_1 + \phi_2 dv_2 = 0, \quad (139)$$

where $\phi_1 = p_1 - e^{q_2} v_1$, $\phi_2 = p_2$. The condition (101) gives $\dot{v}_1 = -v_1 v_2$ as well as the new constraint $\phi_2^{(1)} = v_1 = 0$. Thus the primary constraint ϕ_1 is of the second class. On $M_1 = \{x \in W : \phi_1 = 0\}$ we have

$$\dot{f} = v_1 \frac{\partial f}{\partial q_1} + v_2 \frac{\partial f}{\partial q_2} + \frac{1}{2} e^{q_2} v_1^2 \frac{\partial f}{\partial p_2} - v_1 v_2 \frac{\partial f}{\partial v_1} + \dot{v}_2 \frac{\partial f}{\partial v_2}, \quad (140)$$

$$\phi_2 = p_2 = 0, \quad (141)$$

$$\phi_2^{(1)} = v_1 = 0. \quad (142)$$

Since $\dot{\phi}_2^{(1)} \sim \phi_2^{(1)}$, the acceleration \dot{v}_2 remains undetermined. The constraints ϕ_2 and $\phi_2^{(1)}$ are of the first class. We have also

$$E|_{M_1} = \frac{1}{2} p_1^2 e^{-q_2} + p_2 v_2 = h + v_2 \phi_2, \quad (143)$$

and the system (140) -(142) in Hamiltonian form reads

$$\dot{f} = \{h, f\} + v_2 \{\phi_2, f\} + \dot{v}_2 \frac{\partial f}{\partial v_2}, \quad (144)$$

$$\phi_2 = p_2 = 0, \quad (145)$$

$$\phi_2^{(1)} = p_1 = 0. \quad (146)$$

Once again, the acceleration \dot{v}_2 remains arbitrary.

Example 5: Finally, consider the Lagrangian (Ref. [17])

$$l = \frac{1}{2}v_1^2 + q_2v_1 + (1 - \alpha)q_1v_2 + \frac{\beta}{2}(q_1 - q_2)^2$$

The form $\omega = dp_i \wedge dq_i$ is degenerate on $W = TR^2 \oplus T^*R^2$, $\text{Ker } \omega = \text{span } \{\partial/\partial v_1, \partial/\partial v_2\}$.

The energy

$$E = p_1v_1 + p_2v_2 - l \quad (147)$$

gives the evolution vector field

$$X_E = v_1 \frac{\partial}{\partial q_1} + v_2 \frac{\partial}{\partial q_2} + (v_2(1 - \alpha) + \beta(q_1 - q_2)) \frac{\partial}{\partial p_1} + (v_1 - \beta(q_1 - q_2)) \frac{\partial}{\partial p_2} + \dot{v}_1 \frac{\partial}{\partial v_1} + \dot{v}_2 \frac{\partial}{\partial v_2} \quad (148)$$

as well as the primary constraint form

$$\omega_E = \phi_1 dv_1 + \phi_2 dv_2, \quad (149)$$

where $\phi_1 = p_1 - q_2 - v_1$, $\phi_2 = p_2 - (1 - \alpha)q_1$. For simplicity we consider here only some particular cases.

Case A: $\alpha = 0, \beta = 0$.

The condition (101) gives $\dot{v}_1 = 0$, so ϕ_1 is the primary constraint of the second class. On $M_1 = \{x \in W : \phi_1 = 0\}$ we get

$$X_{E_1} = v_1 \frac{\partial}{\partial q_1} + v_2 \frac{\partial}{\partial q_2} + v_1 \frac{\partial}{\partial p_2} + \dot{v}_2 \frac{\partial}{\partial v_2}. \quad (150)$$

The acceleration \dot{v}_2 cannot be determined. The dynamics is controlled by

$$\dot{f} = X_{E_1}f, \quad (151)$$

$$\phi_2 = p_2 - q_1 = 0. \quad (152)$$

Since

$$E|_{M_1} = (v_1 + q_2)v_1 + p_2v_2 - l = \frac{1}{2}(p_1 - q_2)^2 + v_2\phi_2 = h + v_2\phi_2, \quad (153)$$

the Hamiltonian analog of (151) -(152) is

$$\dot{f} = \{h, f\} + v_2 \{\phi_2, f\} + \dot{v}_2 \frac{\partial f}{\partial v_2}, \quad (154)$$

$$\phi_2 = p_2 - q_1 = 0. \quad (155)$$

Case B: $\alpha = 0, \beta \neq 0$.

In this case the compatibility condition allows to determine $\dot{v}_1 = \beta(q_1 - q_2)$ and gives the secondary constraint $\phi_2^{(1)} = q_2 - q_1$. The primary constraint ϕ_1 is of the second class. On $M_1 = \{x \in W : \phi_1 = 0\}$ the evolution is given by

$$\dot{f} = v_1 \frac{\partial f}{\partial q_1} + v_2 \frac{\partial f}{\partial q_2} + (v_1 - \beta(q_1 - q_2)) \frac{\partial f}{\partial p_2} + \beta(q_1 - q_2) \frac{\partial f}{\partial v_1} + \dot{v}_2 \frac{\partial f}{\partial v_2}, \quad (156)$$

$$\phi_2 = p_2 - q_1 = 0, \quad (157)$$

$$\phi_2^{(1)} = q_2 - q_1 = 0. \quad (158)$$

The next iteration produces $\phi_2^{(2)} = v_2 - v_1$ and the next one gives $\dot{v}_2 = \beta(q_1 - q_2)$. As a result, on $S_3 \subset M_1$, $S_3 = \{x \in M_1 : \phi_2 = 0, \phi_2^{(1)} = 0, \phi_2^{(2)} = 0\}$, we have $\dot{f} = v_1 \frac{\partial f}{\partial q_1}$ and $E|_{S_3} = \frac{1}{2}v_1^2$. To work out the Hamiltonian scheme, note that

$$E|_{M_1} = \frac{1}{2}(p_1 - q_2)^2 - \frac{\beta}{2}(q_1 - q_2)^2 + v_2 \phi_2 = h + v_2 \phi_2. \quad (159)$$

The equation (145) then reads

$$\dot{f} = \{h, f\} + v_2 \{\phi_2, f\} + \dot{v}_2 \frac{\partial f}{\partial v_2}. \quad (160)$$

Again, $\dot{\phi}_2^{(1)} = v_2 - v_1 = \phi_2^{(2)}$ and $\dot{\phi}_2^{(2)} = \dot{v}_2 = 0$. Since $v_2 = -\{h, \phi_2^{(1)}\}$, we can write (149) as

$$\dot{f} = \{h, f\} - \{h, \phi_2^{(1)}\} \{\phi_2, f\} + \{h, \phi_2\} \{\phi_2^{(1)}, f\} \quad (161)$$

(we have added the last term "by hand" to antisymmetrize the bracket, since $\{h, \phi_2\} \sim \phi_2^{(1)} = 0$ on the surface of the second class constraints. The canonical transformation of the form $P_1 = p_1 + p_2 - q_1 - q_2$, $Q_1 = q_1$, $P_2 = p_2 - q_1 = \phi_2$, $Q_2 = q_2 - q_1 = \phi_2^{(1)}$ allows to represent the dynamics on $S_3 = \{P_2 = 0, Q_2 = 0, \phi_2^{(2)} = 0\}$ in a simple way, $\dot{f} = \{h, f\}_{P_1, Q_1}$, where $h = P_1^2/2$.

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APPENDIX: THE GLOSSARY

For convenience of the reader we give some notations and definitions of geometric objects used throughout the paper. More detailed information can be found in Ref. [11, 23].

1. M (or Q) denotes finite-dimensional real smooth manifold, (TM, π_1, M) – its tangent bundle with local coordinates $\{q^i, v^i\}$, (T^*M, π_2, M) – its cotangent bundle with local coordinates $\{q^i, p_i\}$, $i = 1, \dots, N = \dim M$.
2. $\mathcal{X}(M)$ denotes the Lie algebra of smooth vector fields on M . Locally $X \in \mathcal{X}(M)$ is $X = X^i(q) \frac{\partial}{\partial q^i}$, $X^i(q) \in C^\infty(M)$.
3. Correspondingly, $Y \in \mathcal{X}(TM)$ in local coordinates reads $Y = Y^i(q, v) \frac{\partial}{\partial q^i} + V^i(q, v) \frac{\partial}{\partial v^i}$, where Y^i and V^i are smooth functions on TM .
4. $\mathcal{X}^V(TM) = \{X \in \mathcal{X}(TM) : \pi_* X = 0\}$ is the algebra of vertical vector fields on TM . Any $X \in \mathcal{X}^V(TM)$ can be locally represented as $X = X^i(q, v) \frac{\partial}{\partial v^i}$.
5. Lift $\gamma: \mathcal{X}(M) \rightarrow \mathcal{X}^V(TM)$ maps $X = X^i(q) \frac{\partial}{\partial q^i} \in \mathcal{X}(M)$ to $X^V = X^i(q) \frac{\partial}{\partial v^i} \in \mathcal{X}^V(TM)$, where X^V is the generator of the one-parameter group of the fiber diffeomorphisms $t \rightarrow (q^i, v^i + X^i(q)t)$.
6. The canonical (1,1) tensor τ on TM is defined as the composition $\tau = \gamma \circ \pi_*$. In coordinates we have $\tau = dq^i \otimes \frac{\partial}{\partial v^i}$.
7. Cartan – Liouville one-form is defined for $f \in C^\infty(TM)$ by $\theta_f = df \circ \gamma$. If $\Gamma_{ij} = \frac{\partial^2 f}{\partial v^i \partial v^j}$ has maximal rank, $\omega_f = d\theta_f$ defines the symplectic structure on TM .

8. Liouville vector field $\Delta \in \mathcal{X}^V(TM)$ is the generator of the one-parameter group of dilations $(q, v) \rightarrow (q, e^t v)$. Locally, $\Delta = v^i \frac{\partial}{\partial v^i}$.
9. Whitney sum $E_1 \oplus E_2$ of two bundles E_1 and E_2 with the same base B is the bundle with the base B and the fiber $E_{1x} \oplus E_{2x} \forall x \in B$.

References

- [1] P.A.M. Dirac, Canad.J.Math. 2, 129 (1950).
- [2] P.G. Bergmann, Helv.Phys.Acta Suppl.IV 79 (1956).
- [3] L.D.Faddeev, Theor.Math.Phys. 1, No.1, 1 (1969).
- [4] A.M. Vershik and L.D. Faddeev, Soviet Physics –Doklady, 17(1), 34 (1972).
- [5] M.J. Gotay, J.M. Nester and G. Hinds, J.Math.Phys., 19, 2388 (1978).
- [6] M.J. Gotay, J.M. Nester, Ann.Inst.H.Poincaré A30, 129 (1979).
- [7] M.J. Gotay, J.M. Nester, Ann.Inst.H.Poincaré A32, 1 (1980).
- [8] G.Marmo, N.Mukunda and J.Samuel, Rivista Nuovo Cimento 6, 1 (1983).
- [9] J.F. Cariñena, J.Gomis, L.A.Ibort, and N.Román, J.Math.Phys. 26, 1961 (1985).
- [10] C.Battle, J.Gomis, J.M.Pons, and N.Román, J.Math. Phys. 27, 2953 (1986).
- [11] G.Marmo, E.J.Saletan, A.Simoni, and B.Vitale, *Dynamical Systems, A Differential Geometric Approach to Symmetry and Reduction* (Wiley, Chichester, 1985).
- [12] J.F.Cariñena, C.Lopez,, and N. Roman-Roy, J.Math.Phys. 29, 1143 (1988).
- [13] D.M. Gitman and I.V. Tyutin, "Quantization of fields with constraints" (Springer, Berlin, 1990).

- [14] V.V.Nesterenko and A.M.Chervyakov, Theor.Math.Phys. 64, 701 (1985).
- [15] V.P.Pavlov, Theor.Math.Phys. 92, No.3, 1020 (1992).
- [16] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, Princeton, New Jersey, 1992).
- [17] K.Sundermeyer, “Constrained dynamics”, in *Lecture Notes in Physics*, No.169 (Springer, Berlin, 1982).
- [18] A.A.Kirillov, “*Elements of the Theory of Representations*” (Springer, New York, 1976).
- [19] N. Woodhouse, “*Geometric quantization*” (Clarendon Press, Oxford, 1980).
- [20] M.Crampin, J.Phys.A:Math.Gen. 16, 3755 (1983).
- [21] F. Cantrijn, J.F. Cariñena, M.Crampin, and L.A.Ibort, J.Geom. Phys. 3, 353 (1986).
- [22] Iu.I.Manin, *Gauge fields and complex geometry*(Springer, Berlin, 1988).
- [23] G.Morandi, C.Ferrario, G.Lo Vecchio, G.Marmo and C.Rubano, Phys.Repts. 188, 149 (1990).
- [24] O.Krupková, J.Math.Phys., 35, 6557 (1994).
- [25] W. Kundt, *Canonical Quantization of Gauge Invariant Field Theories*, Springer Tracts in Modern Physics, 40, 107 (1966).
- [26] R. Skinner and R. Rusk, J.Math.Phys., 24, 2589 (1983).